

Relaxation of the cosmological constant in a movable brane world

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Abstract

We present numerical evidence that a domain wall in a background with varying vacuum energy density acquires velocity in the direction of decreasing Hubble parameter. This should lead to at least a partial relaxation of the cosmological constant on the wall.

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Brane-world scenarios, according to which we live on a domain wall in a higher-dimensional space [1], open new avenues for thinking about the cosmological constant. Here, we want to explore feasibility of one particular mechanism that may help the “constant” to relax to a small value.

The effective cosmological constant seen by an observer on the wall is determined by the local value of the Hubble parameter, H . It has long been known [2] that a domain wall in a space with *zero* vacuum energy will inflate, i.e. H will be nonzero. To achieve $H = 0$, one may introduce a *negative* vacuum energy in the bulk, with a value tuned precisely to compensate for the effect of the wall tension [3]. Alternatively, one can introduce a scalar field with a dilaton-like coupling to the brane, to obtain static solutions for more generic values of the tension [4]. Or, with more than two external dimensions, one may keep the bulk vacuum energy zero but assume that gravity in the bulk is so weak that the Hubble parameter induced by the wall tension is small enough to be compatible with observations [5].

Here, we want to consider a different idea. If the bulk vacuum energy varies along an extra dimension, the wall may be moving in such a way that the Hubble parameter on it starts at a large positive value but then approaches zero—or even crosses zero and becomes negative. In this way, the wall will move towards a region where two large contributions to H (one from the bulk energy, and the other from the wall tension) nearly cancel. In other words, a finely tuned solution similar to that of ref. [3] will be reached automatically in the course of the evolution. If the cancellation

persists for some time, it may be possible to identify that period of time with the standard cosmology.*

How closely H can get to zero in this scenario will depend on the complicated dynamics occurring when the above two contributions become of the same order. In this paper, we address an earlier, simpler stage of the evolution, when the effect of the wall tension is still much smaller than the effect of the bulk vacuum energy. We have tried to obtain numerically geometries in which H changes little with time but varies significantly in space, and see which way a test domain wall will move in such a background. Our main result is that, in the backgrounds we were able to obtain, the wall quite generically acquires velocity in the direction of *decreasing* Hubble parameter, i.e. towards regions that are “less inflationary”. Although the actual displacements that we have been able to detect in the lifetime of our simulation are quite small, it is encouraging that the final velocity is a sizable fraction of the speed of light.

A suitable smooth background can be formed, for example, by another domain wall, which is much thicker than ours. The difference in the thickness may be due to a difference in masses, or simply in initial profiles, of the scalar fields forming the walls. Towards the end of the paper, we will speculate that there is also a more “economical” possibility, wherein a configuration of one domain wall in the background of another is realized with a single field in the context of topological inflation [6]. For most of the paper, though, we will prefer to think about these two walls as being due to two different fields.

In the remainder of the paper, we (i) describe some a priori guesses about geometries induced by a space-dependent vacuum energy and (ii) present a sample of the numerical results.

We prefer to use isotropic coordinates, in which the metric has the form

$$ds^2 = -N^2(z, t)dt^2 + \psi^4(z, t)[dz^2 + d\mathbf{x}^2] ; \quad (1)$$

N is called lapse, and ψ —conformal factor. This choice of coordinates was adopted from our numerical studies of post-inflationary universe [7], and the numerical algorithm we use here follows closely the one used in that work. We allow for the total of $D > 1$ spatial coordinates, but the metric depends only on one of them—the fifth coordinate z . Thus, $d\mathbf{x}^2$ is the flat Euclidean metric in the remaining $(D - 1)$ dimensional space.

For a preliminary discussion, it will suffice to use the energy constraint

$$\frac{\kappa^2 C_E}{D - 1} \equiv \frac{\kappa^2 E}{D - 1} - \frac{1}{2}DH^2 + 2\frac{\psi''}{\psi^5} + 2(D - 3)\frac{(\psi')^2}{\psi^6} = 0 , \quad (2)$$

*Similarly, H can change sign if the wall itself is static but the bulk vacuum energy or the wall tension change in time. In this case, however, it may be more difficult to make the time when H is close to zero long enough to include the entire known cosmological history.

and the equation for the lapse

$$\frac{N''}{N} - \frac{4\psi'N'}{\psi N} + 2(D-2)\frac{\psi''}{\psi} - 6(D-2)\left(\frac{\psi'}{\psi}\right)^2 = -\kappa^2 \sum_n (\phi'_n)^2. \quad (3)$$

Here E is the energy density (defined below), and

$$H = \frac{2\dot{\psi}}{N\psi} \quad (4)$$

is the local Hubble parameter. The sum on the right-hand side of (3) is over all scalar fields ϕ_n of the model; a prime denotes derivative with respect to z , and a dot—with respect to t ; $\kappa^2 = 8\pi G$; G is bulk Newton's constant.

We first look for purely vacuum solutions with a constant energy density. In this case, the right-hand side of (3) is zero, while E in (2) equals the cosmological constant: $E = \Lambda$. The equations have the following solution:

$$\psi = \frac{A}{\sqrt{z - \alpha t}}, \quad (5)$$

$$N = \psi^2, \quad (6)$$

where A and α are constants. The Hubble parameter (4) is constant: $H = \alpha/A^2$, so whenever $\alpha > 0$ the solution is inflationary. Note that for $\alpha > 0$, the solution is well-defined in the half-space $z > 0$ for any $t < 0$. This is appropriate, since according to (6), t is the conformal time, which for inflationary solutions is limited to negative values.

Eq. (3) is satisfied by (5)–(6) identically, but the energy constraint (2) imposes a relation between the parameters:

$$\alpha^2 - 1 = \frac{2\kappa^2\Lambda}{D(D-1)}A^4. \quad (7)$$

This is the same relation as obtained for a class of solutions in ref. [8], so ours should be the same solutions except written in different coordinates. In particular, for $\Lambda = 0$, we have $\alpha = \pm 1$ and any A . (A will be determined by the junction condition on the wall: e.g. for a static thin wall with the geometry symmetrical about it, $A^{-2} = \kappa^2\sigma/2(D-1)$, where σ is the wall tension.) The $\alpha = 1$ solution is the inflationary solution of ref. [2] in different coordinates.[†] To obtain $\alpha = 0$ (and consequently $H = 0$), we need to have $\Lambda = -D(D-1)/2\kappa^2A^4$. This is the anti-de Sitter solution of ref. [3].

Now, when Λ is not exactly a constant but is slowly varying, we expect the geometry to interpolate between faster expanding regions with larger Λ and slower

[†]The explicit relation between our coordinates and those used in ref. [2] and called here \tilde{z} and \tilde{t} is: $z = \exp(-\tilde{t}) \sinh \tilde{z}$, $t = -\exp(-\tilde{t}) \cosh \tilde{z}$, all in units of the constant Hubble distance H^{-1} .

expanding, or perhaps even contracting, regions with smaller Λ . Geometries interpolating between de Sitter and anti-de Sitter spaces can presumably be obtained by analytical continuation of certain Euclidean instantons. Such instantons, containing a de Sitter region sandwiched between two anti-de Sitter regions, have been considered in ref. [9] in connection with the problem of counting de Sitter microstates. Note that for every positive α (expanding) solution, eq. (7) has a negative α (contracting) solution. Such contracting regions are ubiquitous in our numerical simulations. We do not know, however, if they can be obtained by analytical continuation from any instantons.

To study real-time evolution that may possibly lead to such interpolating geometries, we will need the rest of Einstein's equation. As in ref. [7], we write these in the Hamiltonian form. The Hamiltonian pair to eq. (4) is

$$\frac{\dot{H}}{N} = -\frac{D}{2}H^2 + 2(D-2)\frac{(\psi')^2}{\psi^6} + \frac{2\psi'N'}{\psi^5N} + \frac{\kappa^2}{D-1}(2V - E + wC_E) . \quad (8)$$

(Although we did not emphasize that above, for a constant $E = V = \Lambda$ eq. (8) is also satisfied by the solution (5)–(6).) Equations for the fields are

$$\dot{\phi}_n = N \frac{\pi_n}{\psi^{2D}} , \quad (9)$$

$$\dot{\pi}_n = \left(\psi^{2D-4} N \phi'_n \right)' - N \psi^{2D} \frac{\partial V}{\partial \phi_n} , \quad (10)$$

The energy density is

$$E = \sum_n \left(\frac{\pi_n^2}{2\psi^{4D}} + \frac{(\phi'_n)^2}{2\psi^4} \right) + V(\phi) ; \quad (11)$$

V is the potential of the fields. The energy constraint C_E can be added to the right-hand side of (8) with any coefficient w (however, for numerical studies some choices may be better than others [7]). In what follows, we use $w = 1$.

The Hamiltonian structure of the equations allows us to use a leap-frog algorithm, in which the coordinates ϕ_n and ψ are defined at full time steps, while the momenta π_n and H are defined at half-steps. The lapse does not have a Hamiltonian pair; as in [7], we update it at full steps using eq. (3) and extrapolate it to half-steps. The energy constraint (2) is not used for the evolution but is monitored to provide an idea of how accurate the results are. In addition, we have monitored the momentum constraint

$$\kappa^2 \psi^2 C_M \equiv \frac{\kappa^2}{\psi^{2D}} \sum_n \pi_n \phi'_n + (D-1)H' = 0 \quad (12)$$

and have found that for the time intervals shown in the plots both C_E and C_M stay close to zero to a satisfactory degree.

To start numerical evolution, we need to solve the constraints (2) and (12) at the initial time slice. The momentum constraint (12) is solved trivially with $H = \text{const}$ and the initial choice of the fields such that for each n either π_n or ϕ'_n vanish. There is some advantage in being able to choose a suitable initial profile of ψ by hand. To make our choice compatible with the energy constraint (2), we introduce a massive “radiation” field which is initially zero and whose initial momentum is determined by solving eq. (2). (This is similar to the method used in ref. [10] to study the onset of inflation in inhomogeneous spacetimes.) We call this field “radiation” only to distinguish it from the fields forming the domain walls; its mass is in fact the largest mass in the problem. The “radiation” energy redshifts away in the course of the evolution (of course, more so in regions that are inflationary).

We thus have three fields in our simulation: ϕ_0 that forms the thick background wall, ϕ_1 that forms the movable thin wall, and ϕ_2 , the massive “radiation” field. We keep the thick wall centered at $z = 0$ by choosing the boundary condition $\phi_0(0, t) = 0$. For the potential, we use

$$V = \sum_{n=0,1} \left(-\frac{1}{2}\mu_n^2\phi_n^2 + \frac{1}{4}\lambda_n\phi_n^4 \right) + \frac{\mu_1^4}{4\lambda_1} + \frac{1}{2}M^2\phi_2^2 + \Lambda, \quad (13)$$

where Λ is a constant—it is the vacuum energy at $z = 0$. A nontrivial z -dependence of vacuum energy is due to variation of the field ϕ_0 . With potential (13), the only interaction between different fields is gravitational.

A word about the choice of parameters. We measure times and lengths in units of μ_0^{-1} and energy densities in units of μ_0^4/λ_0 , so that we can set $\mu_0 = 1$ and $\lambda_0 = 1$. The results below are for $\mu_1 = 2$, $\lambda_1 = 2000$, $M = 250$, $\kappa^2 = 37.5$ (this dimensionless κ^2 is $\kappa^2\mu_0^2/\lambda_0$ in the original units), $\Lambda = 0.16$, and $D = 4$ (i.e. one extra dimension).

We want the background to be sufficiently smooth, so that the interior of the thick wall—a region near $z = 0$ —can inflate. This type of inflation is sometimes called topological inflation [6]. So, we choose the initial profile of ϕ_0 in the form

$$\phi_0(z, t = 0) = \frac{z}{\sqrt{z^2 + \ell^2}}, \quad (14)$$

with the spatial scale ℓ much larger than the wall’s “natural” thickness: $\psi^2\ell \gg \mu_0^{-1}$. The plots below are for $\ell = L = 50$, where L is the total length of the integration region: $z \in [0, L]$, from a grid with the total of 2049 points.

Another necessary condition for topological inflation is that near $z = 0$ the Hubble parameter supported by the vacuum energy of ϕ_0 is large enough, in comparison with μ_0 [6]. With the above values of the parameters, this Hubble parameter is $H_0 = (\kappa^2\Lambda/6)^{1/2} = 1$. There have been numerical determinations of the critical ratios H_0/μ_0 for which the interiors of various types of topological defects will inflate in three spatial dimensions [11]. We have not attempted a similarly detailed study for $D = 4$, but have observed large amounts of inflation for the above value of $H_0/\mu_0 = 1$.

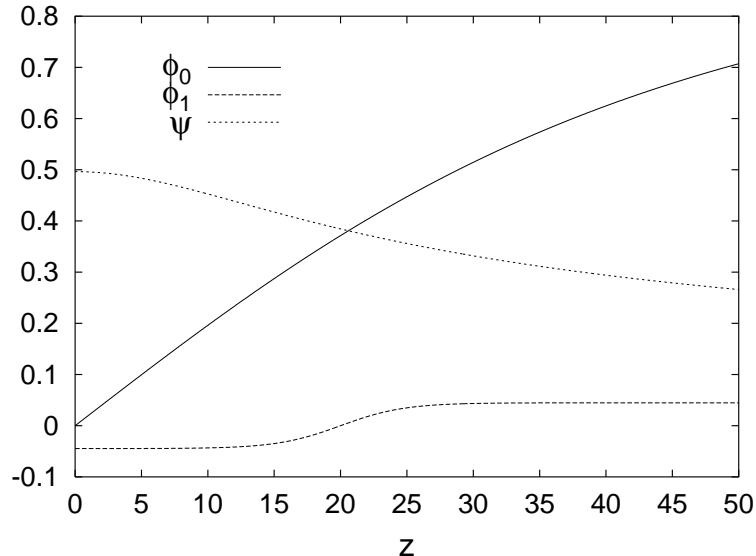


Figure 1: Initial profiles of the two domain walls and of the conformal factor ψ .

In comparison, gravitational effects due to the wall formed by ϕ_1 are, by our choice of the parameters, much smaller. A convenient measure of these effects is the expansion rate the wall *would* have if it were inflating in empty space with zero bulk cosmological constant: $H_1 = \kappa^2 \sigma_1 / 2(D-1)$, where $\sigma_1 \sim \mu_1^3 / \lambda_1$ is the wall tension. For the above values of the parameters, this gives $\sigma_1 \sim 4 \times 10^{-3}$. This estimate is well born out numerically,[‡] so we use $H_1 = 0.03$. Because $H_1 \ll H_0$, the wall formed by ϕ_1 has only a minor effect on the geometry in the rapidly expanding region near $z = 0$. On the other hand, when the wall is in a region where the expansion rate is of order H_1 , or smaller, we expect strong gravitational effects due to the wall's tension. As we have already mentioned, such effects are likely to determine if the present scenario leads to a complete solution to the cosmological constant problem, but they lie outside the scope of this initial study.

In Fig. 1, we show the initial profiles of ϕ_0 , ϕ_1 , and ψ for a typical initial position of the second wall, $z_1 = 20$.

Now, let us see if an interpolating geometry of interest to us is reached in the course of numerical evolution. Evidence for that is shown in Fig. 2, where we plot profiles of the Hubble parameter H at several moments of time. We see that, after a transient behavior, H acquires a nearly static profile, in which it has positive values near $z = 0$ (where the vacuum energy is the largest) and negative values at large z .

To estimate the amount of inflation we obtain near $z = 0$, we plot in Fig. 3 the value of the conformal factor ψ at $z = 0$ as a function of time. We see an exponential

[‡]Variation of the tension with time is relatively small, which implies that the wall formed by ϕ_1 maintains more or less constant physical (as opposed to comoving) thickness.

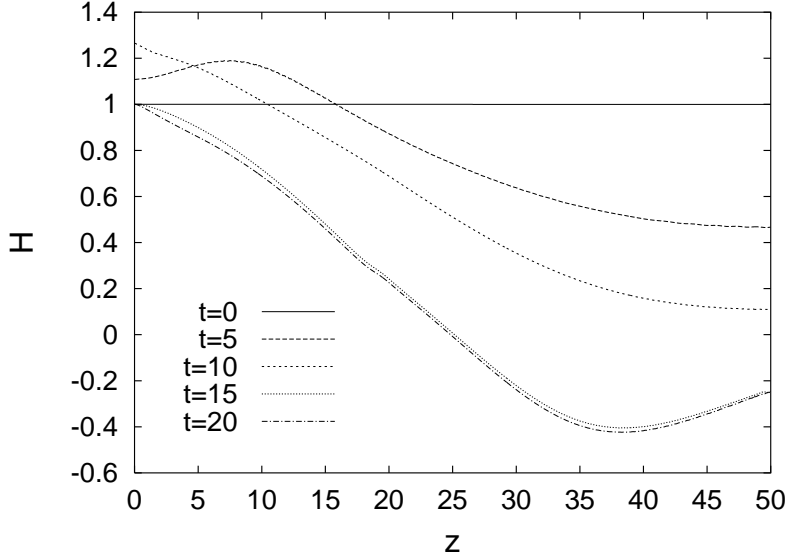


Figure 2: Profiles of the local Hubble parameter at several moments of time.

growth by a factor over 40 between $t = 12.5$ and $t = 20$. This corresponds to the size of the universe growing by a factor over 1600.[§]

We define velocity of the domain wall formed by ϕ_1 as the ratio of the momentum of ϕ_1 to its energy, both taken per unit transverse area:

$$v = -\frac{\int \pi_1 \phi_1' \psi^{-2D} dz}{\int E_1 \psi^2 dz}, \quad (15)$$

where

$$E_1 = \frac{\pi_1^2}{2\psi^{4D}} + \frac{(\phi_1')^2}{2\psi^4} + \frac{1}{4}\lambda_1 \left(\phi_1^2 - \frac{\mu_1^2}{\lambda_1} \right)^2. \quad (16)$$

(The denominator in (15) divided by the γ -factor of the wall is our definition of the wall tension σ_1 .) Note that if the profile of ϕ_1 with respect to the *comoving* coordinate z does not change, i.e. $\partial_t \phi_1 = 0$, the definition (15) gives zero. In other words, v as defined by (15) is insensitive to a motion that is due solely to the expansion of the universe: it measures velocity of the wall relative to the expanding background.

In Fig. 4 we plot the wall's velocity as a function of time, for the same initial position of the wall $z_1 = 20$ that was used for the previous plots, as well as for two

[§]To properly interpret the plot, one needs to know also the time-dependence of the lapse N at $z = 0$, since it is $N(z, t)dt$ that determines the increment of cosmic time at location z . As it happens, after N reaches the value of 1 at $z = 0$ in our simulation, we keep it equal to 1 there at all subsequent times (which is always possible due to the remaining gauge freedom). For Fig. 3, this resetting of the clock rate took place at $t \approx 12.7$; after that, time t coincides (up to a constant) with the cosmic time at $z = 0$.

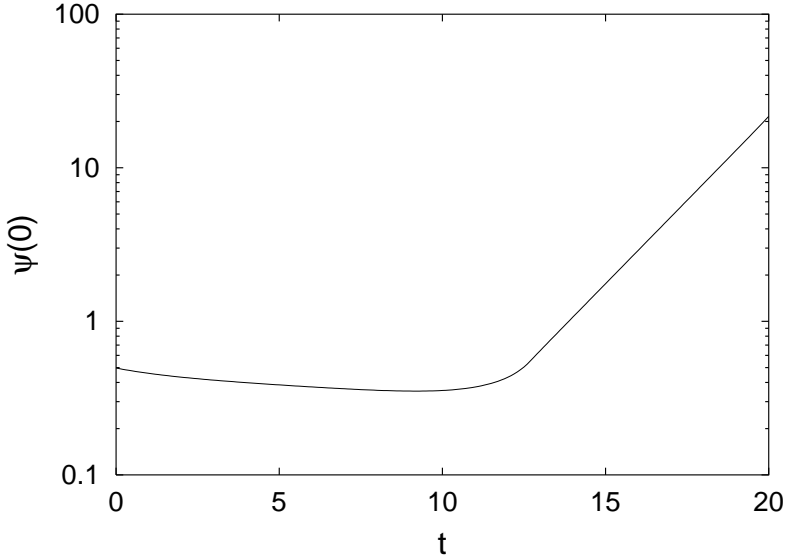


Figure 3: Inflation at the origin: conformal factor at $z = 0$ as a function of time.

smaller values of z_1 . We see that v invariably becomes positive at late times, when a geometry with an almost static profile of the Hubble parameter is reached. The final value of the velocity can be a sizable fraction of the speed of light and is larger for walls that are further away from the origin.

It may appear from Fig. 4 that the velocity goes to a steady asymptotic value, but we should point out that the lapse $N(z, t)$ at the locations of the wall drops precipitously during the late-time evolution. So, the entire stage when v is close to its final value corresponds only to a small interval of cosmic time on the wall. As a consequence, the corresponding displacements of the wall are tiny. On the other hand, one can argue that the velocity remains steady over large amounts of expansion in the inflating region, cf. Fig. 3.

Throughout this paper, we have considered the case when the wall formed by ϕ_1 is much “weaker”, in terms of its gravitational effects, than the wall formed by ϕ_0 . This does not always have to be so, and, as we have already mentioned, perhaps the most interesting case with regard to the cosmological constant problem is when the “strengths” of the two walls are comparable. In addition, walls of different thicknesses but of equal “strengths” naturally arise in the context of topological inflation [6] with an extra dimension. Let us see how.

The wall formed by ϕ_1 in our simulations maintains more or less constant physical thickness, as opposed to typical walls of ϕ_0 considered in topological inflation. Those latter are produced by growth of small fluctuations in the *middle* of the inflating region, where $\phi_0 \approx 0$, and the vacuum energy E_{vac} is much larger than the gradient energy E_{grad} of a new wall (whose thickness at birth is of the order of the Hubble dis-

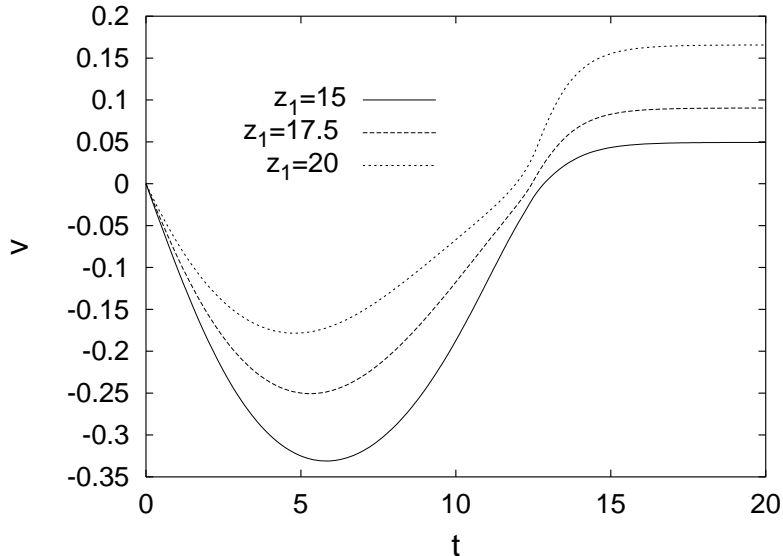


Figure 4: Velocity of the domain wall formed by ϕ_1 as a function of time, for three different values of the initial position.

tance). So, these walls expand in all directions [6]. Analogy with the case considered here is much closer for a wall produced at the *fringe* of the inflating region. If a wall finds itself in a region with $E_{\text{vac}} \sim E_{\text{grad}}$, its thickness will not grow, but its gravitational effects will be comparable to those of the background. This can be seen by comparing the previously introduced parameters H_0 and H_1 . Indeed, $H_0^2 \sim \kappa^2 E_{\text{vac}}$, while $H_1 \sim \kappa^2 \sigma_1 \sim \kappa^2 E_{\text{grad}} H_0^{-1}$, so $H_1 \sim H_0$.

Production of new domains walls of ϕ_0 should be observable even in the model considered here, if we were able to follow the evolution for a long enough time. In fact, it may well be that the eventual breakdown of our simulations (at times larger than those shown in the plots) is due primarily to the onset of this new, more complex dynamics. Obtaining good quality data for such a regime will require further numerical work. We hope to return to this question in a future publication.

In conclusion, we have found that numerically solving Einstein's equations on one-dimensional lattices allows one to obtain, and experiment with, geometries in which the vacuum energy density varies along an extra dimension, and the local value of the Hubble parameter is almost static but varies from point to point. We have also found that a test domain wall in such a geometry will move in the direction of decreasing Hubble parameter, suggesting that this mechanism can lead to at least a partial relaxation of the cosmological “constant” in a brane world.

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